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LETTER TO THE EDITOR

Operator content of the Ashkin–Teller quantum chain—superconformal and Zamolodchikov–Fateev invariance: II. Boundary conditions compatible with the torus

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Abstract. We present the operator content of the Ashkin-Teller quantum chain with eight boundary conditions corresponding to the eight elements of the group D_4 which gives the global symmetry of the system. This operator content is a conjecture based on an extensive numerical study and was proven to be correct at the Ising decoupling point. For the values of the coupling constant where $N=2$ superconformal invariance was seen for free boundary conditions, the situation is now more complex. One finds sectors which have $N=2$ and $N=1$ superconformal invariance and sectors which have no superconformal invariance at all. If, however, we combine the 'wrong' sectors of the spectra which correspond to two different values of the coupling constant, a new symmetry shows up. We also analyse the operator content at the Fateev-Zamolodchikov points. Here the 'wrong' sectors can be described by a few new primary fields.

In the previous letter (Baake *et al* 1987, hereafter denoted by I) we have given the operator content of the finite-size limit spectrum of the Ashkin-Teller model with free boundary conditions. In the present letter we consider the other geometries. Since this letter is the logical continuation of I we will suppose that the reader went through it and we will use the same notations and definitions. As in I, we are going to present here only our main results, as an extended version is going to be published elsewhere.

The Ashkin-Teller model with boundary condition B is

$$H_B = \frac{1-4h}{4\sqrt{\lambda}\, h \sin \pi/4h} \sum_{j=1}^{N} [(\sigma_j + \sigma_j^3 + \varepsilon\sigma_j^2) + \lambda(\Gamma_j\Gamma_{j+1}^3 + \Gamma_j^3\Gamma_{j+1} + \varepsilon\Gamma_j^2\Gamma_{j+1}^2)] \tag{1a}$$

$$(\Gamma_{N+1})^m = B^{mn}(\Gamma_1)^n. \tag{1b}$$

Here N represents the number of sites, the matrices σ_j and Γ_j and the parameters ε, λ and h are defined in I (see (1)-(3)). The matrix B which defines the boundary condition (1b) is one of the eight matrices:

$$\Sigma^l = \begin{pmatrix} i^l & 0 & 0 \\ 0 & i^{2l} & 0 \\ 0 & 0 & i^{3l} \end{pmatrix} \qquad l=0,1,2,3 \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2}$$

and $\Sigma^l C$. These matrices form a reducible representation of the group D_4. In order to find the global symmetry of the Hamiltonian (1a) we consider the linear transformations

$$(\Gamma_j)^m = A^{mn}(\Gamma_j)^n \tag{3}$$

where the matrix A is again one of the matrices Σ^l and $\Sigma^l C$. It is easy to check that H_B is invariant under all the transformations A which commute with B. If $B=1$ (periodic boundary conditions) the symmetry is obviously the whole group D_4. In table 1 we show the symmetry group for each boundary condition.

For a given boundary condition the spectrum of the Hamiltonian separates according to the irreducible representations (IR) of the corresponding symmetry group. These are the *sectors* of the theory. Taking into account that the D_4 group has one two-dimensional representation (D) and four one-dimensional representations $D_{\alpha\beta}$ (α, β = 1, 2) (for the definitions see (I.14) and (I.15)) and that the groups Z_4 and $Z_2 \otimes Z_2$ have each four IR one finds that altogether we have to consider 34 sectors. Fortunately, they are not all independent.

Let now $E_k(P, N)$ be the energy levels within a sector for the Hamiltonian with N sites. Here P denotes the momentum (with our boundary conditions the Hamiltonian is translational invariant) and k labels the level. We denote by $E_0(N)$ the ground-state energy, i.e. the lowest level in the sector $D_{0,0}$ with periodic boundary conditions, and consider the quantities:

$$\mathscr{E}_k(P) = \lim_{N\to\infty} \frac{N}{2\pi}(E_k(P, N) - E_0(N)) \tag{4}$$

which define the finite-size spectrum of the theory.

It us a consequence of conformal invariance in two dimensions that the $\mathscr{E}_k(P)$ can be described in terms of IR of two commuting Virasoro algebras with a central charge $c=1$. Namely, an IR $(\Delta, \bar{\Delta})$ generates the levels

$$\begin{aligned} \mathscr{E}(P) &= \Delta + r + \bar{\Delta} + \bar{r} \\ P &= (\Delta + r) - (\bar{\Delta} + \bar{r}) \end{aligned} \tag{5}$$

with a degeneracy $D(\Delta, r)D(\bar{\Delta}, \bar{r})$ ($D(\Delta, r)$ is given in (I.9) and (I.10)).

After a painful numerical analysis and inspired by the extended scaling relations of the Gauss model (den Nijs 1981, Friedan and Shenker 1986), we have arrived at the following conjecture for the finite-size spectra of all the sectors of the model. There are in all 13 independent sectors. In table 2 we show where they occur. In order to save space we have presented only the sectors corresponding to five boundary conditions, the sectors corresponding to the remaining three boundary conditions are going to be given in the extended version of this letter. The operator content of each of the 13 independent sectors ($\mathscr{A}$, $\mathscr{B}$, ... $\mathscr{L}$) is described by an infinite number of IR of the Virasoro algebra. Two types of operators $(\Delta, \bar{\Delta})$ occur. Some have anomalous dimensions independent of h and some are dependent on h. The last ones have the

Table 1. Symmetry groups for different boundary conditions.

Boundary condition	Symmetry group	Elements
$\mathbb{1}$	D_4	$\Sigma^l, \Sigma^l C, l = 0, 1, 2, 3$
Σ^2	D_4	$\Sigma^l, \Sigma^l C, l = 0, 1, 2, 3$
Σ, Σ^3	Z_4	$\Sigma^l, l = 0, 1, 2, 3$
$\Sigma^k C$	$Z_2 \otimes Z_2$	$\Sigma^2, \Sigma^k C, \Sigma^{k+2} C, \mathbb{1}$

Table 2. Operator content for the various sectors of the model. A sector is defined by an irreducible representation of the symmetry group corresponding to a given boundary condition.

	Boundary condition			Boundary condition
Sector	$\mathbb{1}$	Σ^2	Sector	Σ
$D_{0,0}$	$\mathscr{A}$	$\mathscr{G}$	$\Sigma = 1$	$\mathscr{H}$
$D_{1,0}$	$\mathscr{C}$	$\mathscr{F}$	$\Sigma = \mathrm{i}$	$\mathscr{L}$
$D_{0,1}$	$\mathscr{G}$	$\mathscr{B}$	$\Sigma = -1$	$\mathscr{H}$
$D_{1,1}$	$\mathscr{F}$	$\mathscr{C}$	$\Sigma = -\mathrm{i}$	$\tilde{\mathscr{L}}$
D	$\mathscr{H}$	$\mathscr{K}$		

	Boundary condition			Boundary condition
Sector	C		Sector	ΣC
$\Sigma^2 = 1, C = 1$	$\mathscr{D}$		$\Sigma^2 = 1, \Sigma C = 1$	$\mathscr{H}$
$\Sigma^2 = 1, C = -1$	$\mathscr{J}$		$\Sigma^2 = 1, \Sigma C = -1$	$\mathscr{K}$
			$\Sigma^2 = -1, \Sigma C = 1$	$\mathscr{J}$
			$\Sigma^2 = -1, \Sigma C = -1$	$\mathscr{E}$

structure:

$$(\Delta, \bar{\Delta}) = \left(\frac{(M+Nh)^2}{16h}, \frac{(M-Nh)^2}{16h}\right) \qquad M, N \in \mathbb{Z}. \tag{6}$$

We now enumerate the operator content of the 13 independent sectors.

(a) Sectors with $P \in \mathbb{Z}$ (integer momentum)

$$\mathscr{A} = (\{0\}, \{0\}) \oplus (\{1\}, \{1\}) \oplus \mathscr{A}_1$$

$$\mathscr{B} = (\{0\}, \{1\}) \oplus (\{1\}, \{0\}) \oplus \mathscr{A}_1$$

$$\mathscr{A}_1 = \bigoplus_{n \geqslant 0} \left(((n+1)^2 h, (n+1)^2 h) \oplus \left(\frac{(n+1)^2}{h}, \frac{(n+1)^2}{h} \right) \right) \oplus R(4,4;4,4|h) \oplus R(4,4;-4,-4|h)$$

$$\mathscr{C} = R(4,2;4,2|h) \oplus R(4,2;-4,-2|h)$$

$$\mathscr{D} = \bigoplus_{n \geqslant 0} \left(\frac{(2n+1)^2}{16} h, \frac{(2n+1)^2}{16} h \right) \oplus R(4,4;2,1|h) \oplus R(4,4;-2,-1|h)$$

$$\mathscr{E}(h) = \mathscr{D}\left(\frac{1}{h}\right)$$

$$\mathscr{F} = \bigoplus_{n \geqslant 0} \left(\frac{(2n+1)^2}{4h}, \frac{(2n+1)^2}{4h} \right) \oplus R(4,2;4,4|h) \oplus R(4,2;-4,-4|h)$$

$$\mathscr{G}(h) = \mathscr{F}\left(\frac{1}{h}\right)$$

$$\mathscr{H} = ([\tfrac{1}{16}]_1, [\tfrac{1}{16}]_1) \oplus ([\tfrac{9}{16}]_1, [\tfrac{9}{16}]_1). \tag{7a}$$

(b) Sectors with $P \in \mathbb{Z}+\frac{1}{2}$ (half-integer momentum)

$$\mathcal{J} = R(4,2;2,1\,|\,h) \oplus R(4,2;-2,-1\,|\,h) \qquad \mathcal{J}(h) = \mathcal{J}\left(\frac{1}{h}\right)$$

$$\mathcal{K} = ([\tfrac{9}{16}]_1, [\tfrac{1}{16}]_1) \oplus ([\tfrac{1}{16}]_1, [\tfrac{9}{16}]_1). \tag{7b}$$

(c) Sector with $P \in \mathbb{Z}+\frac{1}{4}$

$$\mathcal{L} = R(4,1;4,1\,|\,h) \oplus (4,3;4,3\,|\,h) \oplus R(4,1;-4,-3\,|\,h) \oplus R(4,3;-4,-1\,|\,h). \tag{7c}$$

(d) Sectors with $P \in \mathbb{Z}+\frac{3}{4}$

$$\tilde{\mathcal{L}} = R(4,1;-4,-1\,|\,h) \oplus R(4,3;-4,-3\,|\,h) \oplus R(4,1;4,3\,|\,h) \oplus R(4,3;4,1\,|\,h) \tag{7d}$$

where

$$R(p,q;r,s\,|\,h) = \bigoplus_{\substack{m \geq 0 \\ n \geq 0}} \left(\frac{(pm+q+(rn+s)h)^2}{16h}, \frac{(pm+q-(rn+s)h)^2}{16h} \right). \tag{8}$$

$\{0\}$ and $\{1\}$ are defined in (I.24), $[\frac{1}{16}]_1$ and $[\frac{9}{16}]_1$ are defined in (I.28). We notice that, as in the case of free boundary conditions, one has operators which are h dependent and others that are not. It is amusing to note that, combining the two h independent sectors $\mathcal{H}$ and $\mathcal{K}$, one has

$$\mathcal{H} \oplus \mathcal{K} = ((\tfrac{1}{16})_1^{\mathrm{NS}}, (\tfrac{1}{16})_1^{\mathrm{NS}}) \tag{9}$$

where we have again used (I.28). This implies that these two sectors have $N=1$ superconformal invariance for any h (including the Ising decoupling point!). It is also important to note that combining the sectors $\mathcal{A}$ and $\mathcal{B}$ and using (I.25) we deduce that for any h the system is invariant under two commuting U(1) Kac–Moody algebras (see (I.23)). We also notice the relations:

$$\mathcal{A}(h) = \mathcal{A}\left(\frac{1}{h}\right) \qquad \mathcal{B}(h) = \mathcal{B}\left(\frac{1}{h}\right) \qquad \mathcal{C}(h) = \mathcal{C}\left(\frac{1}{h}\right)$$

$$\mathcal{L}(h) = \mathcal{L}\left(\frac{1}{h}\right) \qquad \tilde{\mathcal{L}}(h) = \tilde{\mathcal{L}}\left(\frac{1}{h}\right). \tag{10}$$

This implies that the spectra for the pair of coupling constants h and $1/h$ are the same. This was not the case for free boundary conditions.

We now come to the problem of higher symmetries in the model. We start with superconformal invariance. In I we have noticed that for free boundary conditions the system has $N=2$ superconformal invariance at $h=\frac{1}{6}, \frac{2}{3}, \frac{3}{2}$ and 6. Based on our previous observation let us consider now only the points $h=\frac{2}{3}$ and 6. Using (I.28) we have for both $h=\frac{2}{3}$ and 6

$$\begin{aligned}\mathcal{A} \oplus \mathcal{B} \oplus 2\mathcal{G} = {} & ([0]_1,[0]_1) \oplus ([1]_1,[1]_1) \oplus ([0]_1,[1]_1) \\ & \oplus ([1]_1,[0]_1) \oplus 4([\tfrac{3}{2}]_1,[\tfrac{3}{2}]_1) \oplus 2([\tfrac{2}{3}]_1,[\tfrac{2}{3}]_1) \oplus 2([\tfrac{1}{6}]_1,[\tfrac{1}{6}]_1) \\ \mathcal{J} = {} & ([\tfrac{3}{2}]_1,[0]_1) \oplus ([\tfrac{3}{2}]_1,[1]_1) \oplus ([0]_1,[\tfrac{3}{2}]_1) \\ & \oplus ([1]_1,[\tfrac{3}{2}]_1) \oplus ([\tfrac{2}{3}]_1,[\tfrac{1}{6}]_1) \oplus ([\tfrac{1}{6}]_1,[\tfrac{2}{3}]_1).\end{aligned} \tag{11a}$$

Alternatively,

$$\mathscr{A}\oplus\mathscr{B}\oplus 2\mathscr{G}\oplus 2\mathscr{I}=((0)_1^{\mathrm{NS}},(0)_1^{\mathrm{NS}})\oplus((1)_1^{\mathrm{NS}},(1)_1^{\mathrm{NS}})\oplus((1)_1^{\mathrm{NS}},(0)_1^{\mathrm{NS}})$$
$$\oplus((0)_1^{\mathrm{NS}},(1)_1^{\mathrm{NS}})\oplus 2((\tfrac{1}{6})_1^{\mathrm{NS}},(\tfrac{1}{6})_1^{\mathrm{NS}}) \tag{11b}$$

and

$$\mathscr{C}\oplus\mathscr{D}\oplus\mathscr{F}=((\tfrac{1}{24})_1^{\mathrm{R}},(\tfrac{1}{24})_1^{\mathrm{R}})\oplus 2((\tfrac{3}{8})_1^{\mathrm{R}},(\tfrac{3}{8})_1^{\mathrm{R}}). \tag{12}$$

It is easy to check that (11b) and (12) imply $N=2$ superconformal invariance. The difference between the points $h=\frac{2}{3}$ and 6 lies in the secotrs $\mathscr{E}$, $\mathscr{J}$, $\mathscr{L}$ and $\tilde{\mathscr{L}}$. Their operator content looks hectic at first sight and the anomalous dimensions given by superconformal invariance do not show up at all. We call these sectors the 'wrong' sectors. An interesting phenomenon occurs, however, if we *combine* the corresponding sectors of $h=\frac{2}{3}$ and $h=6$. In order to explain what happens let us define some new fields (ascendant plus descendant) which are expressed in terms of IR of the Virasoro algebra:

$$(\tfrac{3}{32})^{\mathrm{W}}=\bigoplus_{k\in\mathbb{Z}}(\tfrac{3}{32}(16k+1)^2);\ (\tfrac{27}{32})^{\mathrm{W}}=\bigoplus_{k\in\mathbb{Z}}(\tfrac{3}{32}(16k+3)^2)$$
$$(\tfrac{75}{32})^{\mathrm{W}}=\bigoplus_{k\in\mathbb{Z}}(\tfrac{3}{32}(16k+5)^2);\ (\tfrac{147}{32})^{\mathrm{W}}=\bigoplus_{k\in\mathbb{Z}}(\tfrac{3}{32}(16k+7)^2) \tag{13a}$$

and

$$(\tfrac{1}{96})^{\mathrm{W}}=\bigoplus_{k\in\mathbb{Z}}\left(\frac{(48k+1)^2}{96}\right)\oplus\bigoplus_{k\in\mathbb{Z}}\left(\frac{(48k+17)^2}{96}\right)$$
$$(\tfrac{25}{96})_{W}=\bigoplus_{k\in\mathbb{Z}}\left(\frac{(48k+5)^2}{96}\right)\oplus\bigoplus_{k\in\mathbb{Z}}\left(\frac{(48k+11)^2}{96}\right)$$
$$(\tfrac{49}{96})^{\mathrm{W}}=\bigoplus_{k\in\mathbb{Z}}\left(\frac{(48k+7)^2}{96}\right)\oplus\bigoplus_{k\in\mathbb{Z}}\left(\frac{(48k+23)^2}{96}\right)$$
$$(\tfrac{169}{96})^{\mathrm{W}}=\bigoplus_{k\in\mathbb{Z}}\left(\frac{(48k+13)^2}{96}\right)\oplus\bigoplus_{k\in\mathbb{Z}}\left(\frac{(48k+19)^2}{96}\right). \tag{13b}$$

We can now express the 'wrong' sectors in terms of 'wrong' fields:

$$\mathscr{E}(h=\tfrac{2}{3})\oplus\mathscr{E}(h=6)=((\tfrac{1}{96})^{\mathrm{W}},(\tfrac{1}{96})^{\mathrm{W}})\oplus((\tfrac{25}{96})^{\mathrm{W}},(\tfrac{25}{96})^{\mathrm{W}})\oplus((\tfrac{49}{96})^{\mathrm{W}},(\tfrac{49}{96})^{\mathrm{W}})$$
$$\oplus((\tfrac{169}{96})^{\mathrm{W}},(\tfrac{169}{96})^{\mathrm{W}})\oplus 2((\tfrac{3}{32})^{\mathrm{W}},(\tfrac{3}{32})^{\mathrm{W}})\oplus 2((\tfrac{27}{32})^{\mathrm{W}},(\tfrac{27}{32})^{\mathrm{W}})$$
$$\oplus 2((\tfrac{75}{32})^{\mathrm{W}},(\tfrac{75}{32})^{\mathrm{W}})\oplus 2((\tfrac{147}{32})^{\mathrm{W}},(\tfrac{147}{32})^{\mathrm{W}})$$
$$\mathscr{J}(h=\tfrac{2}{3})\oplus\mathscr{J}(h=6)=((\tfrac{49}{96})^{\mathrm{W}},(\tfrac{1}{96})^{\mathrm{W}})\oplus((\tfrac{169}{96})^{\mathrm{W}},(\tfrac{25}{96})^{\mathrm{W}})\oplus 2((\tfrac{75}{32})^{\mathrm{W}},(\tfrac{27}{32})^{\mathrm{W}})$$
$$\oplus 2((\tfrac{147}{32})^{\mathrm{W}},(\tfrac{3}{32})^{\mathrm{W}})\oplus((\tfrac{1}{96})^{\mathrm{W}},(\tfrac{49}{96})^{\mathrm{W}})\oplus((\tfrac{25}{96})^{\mathrm{W}},(\tfrac{169}{96})^{\mathrm{W}})$$
$$\oplus 2((\tfrac{27}{32})^{\mathrm{W}},(\tfrac{75}{32})^{\mathrm{W}})\oplus 2((\tfrac{27}{32})^{\mathrm{W}},(\tfrac{75}{32})^{\mathrm{W}})\oplus 2((\tfrac{3}{32})^{\mathrm{W}},(\tfrac{147}{32})^{\mathrm{W}})$$
$$\mathscr{L}(\tfrac{2}{3})\oplus\mathscr{L}(6)=((\tfrac{25}{96})^{\mathrm{W}},(\tfrac{1}{96})^{\mathrm{W}})\oplus((\tfrac{49}{96})^{\mathrm{W}},(\tfrac{25}{96})^{\mathrm{W}})\oplus((\tfrac{169}{96})^{\mathrm{W}},(\tfrac{49}{96})^{\mathrm{W}})$$
$$\oplus((\tfrac{1}{96})^{\mathrm{W}},(\tfrac{169}{96})^{\mathrm{W}})\oplus 2((\tfrac{3}{32})^{\mathrm{W}},(\tfrac{27}{32})^{\mathrm{W}})\oplus 2((\tfrac{27}{32})^{\mathrm{W}},(\tfrac{147}{32})^{\mathrm{W}})$$
$$\oplus 2((\tfrac{147}{32})^{\mathrm{W}},(\tfrac{75}{32})^{\mathrm{W}})\oplus 2((\tfrac{75}{32})^{\mathrm{W}},(\tfrac{3}{32})^{\mathrm{W}}). \tag{14}$$

$\tilde{\mathscr{L}}(\frac{2}{3})\oplus\tilde{\mathscr{L}}(6)$ can be obtained from $\mathscr{L}(\frac{2}{3})\oplus\mathscr{L}(6)$ changing $(\Delta,\bar{\Delta})$ into $(\bar{\Delta},\Delta)$.

We now notice that the ascendants in (13*a*) and in (13*b*) are related in a simple way:

$$\begin{aligned} &\tfrac{27}{32}=\tfrac{3}{32}+\tfrac{3}{4} \qquad \tfrac{75}{32}=\tfrac{3}{32}+\tfrac{9}{4} \qquad \tfrac{147}{32}=\tfrac{3}{32}+\tfrac{9}{2} \\ &\tfrac{25}{96}=\tfrac{1}{96}+\tfrac{1}{4} \qquad \tfrac{49}{96}=\tfrac{1}{96}+\tfrac{1}{2} \qquad \tfrac{169}{96}=\tfrac{1}{96}+\tfrac{7}{4} \end{aligned} \tag{15}$$

which suggests the introduction of two *multiplet fields*:

$$\begin{aligned} &(\tfrac{3}{32})^{\mathrm{W}}_{\mathrm{M}}=(\tfrac{3}{32})^{\mathrm{W}}\oplus(\tfrac{27}{32})^{\mathrm{W}}\oplus(\tfrac{75}{32})^{\mathrm{W}}\oplus(\tfrac{147}{32})^{\mathrm{W}} \\ &(\tfrac{1}{96})^{\mathrm{W}}_{\mathrm{M}}=(\tfrac{1}{96})^{\mathrm{W}}\oplus(\tfrac{25}{96})^{\mathrm{W}}\oplus(\tfrac{49}{96})^{\mathrm{W}}\oplus(\tfrac{169}{96})^{\mathrm{W}}. \end{aligned} \tag{16}$$

In terms of the multiplet fields the operator content of all the 'wrong' sectors has a very simple expression:

$$\begin{aligned} &\mathscr{E}(\tfrac{2}{3})\oplus\mathscr{J}(\tfrac{2}{3})\oplus\mathscr{L}(\tfrac{2}{3})\oplus\tilde{\mathscr{L}}(\tfrac{2}{3})\oplus\mathscr{E}(6)\oplus\mathscr{J}(6)\oplus\mathscr{L}(6)\oplus\tilde{\mathscr{L}}(6) \\ &\qquad =((\tfrac{1}{96})^{\mathrm{W}}_{\mathrm{M}},(\tfrac{1}{96})^{\mathrm{W}}_{\mathrm{M}})\oplus 2((\tfrac{3}{32})^{\mathrm{W}}_{\mathrm{M}},(\tfrac{3}{32})^{\mathrm{W}}_{\mathrm{M}}). \end{aligned} \tag{17}$$

Some comments are now in order. The idea of combining sectors that correspond to two different coupling constants is less strange than it looks. A similar situation occurs already in the defected Ising model (Henkel and Patkós 1987). The 'wrong' fields and multiplet fields are IR of some infinite Lie algebra that we do not know. Algebras of this kind were touched upon by Zamolodchikov (1985).

After having discussed the supersymmetric points of the model we now consider the two points $h=\frac{1}{3}$ and 3 which have Zamolodchikov-Fateev (1985) invariance. As discussed earlier, although the operator content for these two values of h is different for free boundary conditions, it is the same for the other geometries. In the present case the 'right' sectors are $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{L}$ and $\tilde{\mathscr{L}}$. We get

$$\begin{aligned} &\mathscr{A}\oplus\mathscr{B}\oplus 2\mathscr{C}=([0]^{\mathrm{ZF}}_{\mathrm{M}},[0]^{\mathrm{ZF}}_{\mathrm{M}})\oplus 2([\tfrac{1}{3}]^{\mathrm{ZF}},[\tfrac{1}{3}]^{\mathrm{ZF}}) \\ &\mathscr{F}\oplus\mathscr{G}=([\tfrac{1}{12}]^{\mathrm{ZF}},[\tfrac{1}{12}]^{\mathrm{ZF}})\oplus 2([\tfrac{3}{4}]^{\mathrm{ZF}},[\tfrac{3}{4}]^{\mathrm{ZF}}) \\ &\mathscr{L}\oplus\tilde{\mathscr{L}}=([0]^{\mathrm{ZF}}_{\mathrm{M}},[\tfrac{3}{4}]^{\mathrm{ZF}})\oplus([\tfrac{3}{4}]^{\mathrm{ZF}},[0]^{\mathrm{ZF}}_{\mathrm{M}})\oplus([\tfrac{1}{12}]^{\mathrm{ZF}},[\tfrac{1}{3}]^{\mathrm{ZF}})\oplus([\tfrac{1}{3}]^{\mathrm{ZF}},[\tfrac{1}{12}]^{\mathrm{ZF}}) \end{aligned} \tag{18}$$

where

$$[0]^{\mathrm{ZF}}_{\mathrm{M}}=[0]^{\mathrm{ZF}}\oplus[1]^{\mathrm{ZF}}\oplus 2[3]^{\mathrm{ZF}} \tag{19}$$

(we have used here (I.30)). The 'wrong' sectors (where different anomalous dimensions occur) are $\mathscr{D}$, $\mathscr{E}$, $\mathscr{I}$ and $\mathscr{J}$. Their operator content is

$$\begin{aligned} \mathscr{D}\oplus\mathscr{E}&=((\tfrac{1}{48})^{\phi},(\tfrac{1}{48})^{\phi})\oplus((\tfrac{25}{48})^{\phi},(\tfrac{25}{48})^{\phi})\oplus 2((\tfrac{3}{16})^{\phi},(\tfrac{3}{16})^{\phi}) \\ &\quad\oplus 2((\tfrac{27}{16})^{\phi},(\tfrac{27}{16})^{\phi}) \\ \mathscr{I}\oplus\mathscr{J}&=((\tfrac{1}{48})^{\phi},(\tfrac{25}{48})^{\phi})\oplus((\tfrac{25}{48})^{\phi},(\tfrac{1}{48})^{\phi})\oplus 2((\tfrac{3}{16})^{\phi},(\tfrac{27}{16})^{\phi}) \\ &\quad\oplus 2((\tfrac{27}{16})^{\phi},(\tfrac{3}{16})^{\phi}) \end{aligned} \tag{20}$$

where

$$\begin{aligned} (\tfrac{1}{48})^{\phi}&=\bigoplus_{k\in\mathbf{Z}}\left(\frac{(24k+1)^2}{48}\right)\oplus\bigoplus_{k\in\mathbf{Z}}\left(\frac{(24k+7)^2}{48}\right) \\ (\tfrac{25}{48})^{\phi}&=\bigoplus_{k\in\mathbf{Z}}\left(\frac{(24k+5)^2}{48}\right)\oplus\bigoplus_{k\in\mathbf{Z}}\left(\frac{(24k+11)^2}{48}\right) \\ (\tfrac{3}{16})^{\phi}&=\bigoplus_{k\in\mathbf{Z}}\left(\frac{3}{16}(8k+1)^2\right) \\ (\tfrac{27}{16})^{\phi}&=\bigoplus_{k\in\mathbf{Z}}\left(\frac{3}{16}(8k+5)^2\right). \end{aligned} \tag{21}$$

This concludes our study of the Ashkin–Teller model. One lesson we have learned in this work is that there is room for an abundance of higher symmetries through an interplay of sectors and coupling constants, superconformal invariance being just one possible choice.

References

Baake M, von Gehlen G and Rittenberg V 1987 *J. Phys. A: Math. Gen.* **20** L479
den Nijs M P M 1981 *Phys. Rev.* B **23** 6111
Friedan D and Shenker S 1986 to be published
Henkel M and Patkós A 1987 *Nucl. Phys.* B to be published
Zamolodchikov A B 1985 *Teor. Mat. Fis.* **65** 347
Zamolodchikov A B and Fateev V A 1985 *Sov. Phys.-JETP* **62** 215